ASYMPTOTIC OF THE ELECTROMAGNETIC FIELD ABOVE A LAYERLIKE ANISOTROPIC MEDIUM OF HIGH CONDUCTIVITY

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UDC 538.566

The model of an anisotropic, layerlike medium is often employed in problems of electromagnetic probing, and many papers have been written on the propagation of an electromagnetic field in such media. A systematic exposition of such problems was set out by Tikhonov, Skugarevskaya, and Dmitriev [1-6]. In this paper we shall construct an asymptotic for the electromagnetic field of a point source lying above a layerlike medium of finite anisotropy having a fairly high longitudinal and transverse conductivity, or in which the source lies at a considerable height above the medium. The principles here laid down for the construction of the asymptotic indicate quite clearly under what circumstances the asymptotic is feasible, and if necessary allow the next approximations to be taken into account.

Let us introduce a rectangular Cartesian spatial coordinate system x_1, x_2, x_3 and consider that the dielectric constant ϵ and conductivity σ depend solely on one coordinate x_3, while the magnetic permeability is $\mu = 1$. The upper half-space ($x_3 > 0$) has constants ϵ, σ and the lower half-space ($x_3 < 0$) variable values of these quantities. Since we are considering anisotropic media in which these characteristics are different in the horizontal and vertical directions, we have $\epsilon_m = \epsilon_m(x_3)$, $\sigma_m = \sigma_m(x_3)$ (m = 1, 2, 3 are the numbers of the Cartesian axes). As regards the functions ϵ_m and σ_m we shall consider that these are continuous in each of the closed segments

$$x_{3p} \leqslant x_3 \leqslant x_{3p-1} \qquad (p = 1,\ 2,\ \ldots,\ n-1),$$

where n is the number of layers of the lower half-space. Above the layerlike medium at a point with coordinates (x_{10}, x_{20}, h, where $h > 0$) is a radiating source of intensity $I_m = \mathcal{J}_m e^{i\omega t}$, where $\mathcal{J}_m$ is the amplitude of the current in the direction m; ω is the angular frequency. The problem consists in finding the electromagnetic field of this radiator. Let us consider the field in the upper half space ($x_3 > 0$). In this system of coordinates for $\zeta > 0$ the components of the vector potential u satisfy three inhomogeneous Helmholtz equations

$$\Delta u_m + k_0^2 u_m = f_m, \tag{1}$$

for $\zeta < 0$ they satisfy the homogeneous Helmholtz equations

$$\Delta u_m + k_m^2 u_m = 0 \tag{2}$$

when m = 1, 2, and an equation of the type

$$\frac{\partial^2 u_3}{\partial \xi^2} + \frac{\partial^2 u_3}{\partial \eta^2} + \frac{1}{\lambda^2} k_t^2 \frac{\partial}{\partial \zeta}\left(\frac{1}{k_t^2}\frac{\partial u_3}{\partial \zeta}\right) + k_3^2 u_3 = \left(1 - \frac{1}{\lambda^2}\right)\frac{\partial}{\partial \zeta}\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial \eta}\right) - \frac{1}{\lambda^2} k_t^2 \frac{\partial}{\partial \zeta}\left(\frac{1}{k_t^2}\right)\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial \eta}\right) \tag{3}$$

when m = 3. Here

$$\xi = x_1/h;\ \eta = x_2/h;\ \ \zeta = x_3/h;$$

$$k_0^2 = (\omega^2 \mu_0 \varepsilon_0 - i\omega\mu_0\sigma_0)\, h^2 = a_0^2 - ib_0^2 = \text{const};$$

$$k_m^2 = (\omega^2 \mu_0 \varepsilon_m - i\omega\mu_0\sigma_m)\, h^2 = a_m^2(\zeta) - ib_m^2(\zeta);$$

Rostov-on-Don. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 175-188, March-April, 1966. Original article submitted May 21, 1974.

$$\lambda^2 = k_t^2/k_3^2,\ k_t^2 = k_3^2 \quad \text{with } m = 1,2;\ \Delta \text{ is the Laplace operator;}$$

$$f_m = f_m(\xi, \eta, \zeta) = -\mu_0 I_m \delta(\xi - \xi_1)\delta(\eta - \eta_1)\delta(\zeta - 1),$$
where $\xi_1 = x_{10}/h;\ \eta_1 = x_{20}/h;\ \delta$ is the Dirac delta function.

At the points of discontinuity of the function k_m^2, when $\zeta = \zeta_p$

$$(\zeta_p = x_{3p}/h)\,(p = 0, 1, 2, \ldots, n-1), \text{ where } \zeta_0 = 0,\ k_m^2(+\zeta_0) = k_0^2,$$

the following "splicing" conditions should be satisfied:

$$[u_m(\zeta_p)] = 0 \quad (m = 1, 2, 3), \quad [\partial u_m(\zeta_p)/\partial\zeta] = 0 \tag{4}$$

$$(m = 1, 2), \left[(1/k_t^2)\operatorname{div} u(\zeta_p)\right] = 0,$$

the square brackets denoting discontinuity of the functions

$$[u(\zeta_p)] = u(\xi, \eta, \zeta)|_{\zeta=\zeta_p+0} - u(\xi, \eta, \zeta)|_{\zeta=\zeta_p-0}.$$

In addition to this, u_m satisfies the conditions of radiation at infinity.

For $\zeta < 0$ we introduce the notation $b^2 = -(\min/\zeta)(\operatorname{Re} k_2^t + \operatorname{Im} k^2)$, $d^2 = k_t^2/b^2$. The parameter $b > 0$ is assumed large. Let us construct the asymptotic of the functions $u_m(\xi, \eta, \zeta)$ with respect to $\epsilon = 1/b$. For this purpose we apply a Fourier transformation in the variables ξ, η to the problem (1)-(4) and obtain the following:

for $\zeta > 0$

$$\frac{\partial^2 \bar{\bar{u}}_m}{\partial\zeta^2} - q_0^2 \bar{\bar{u}}_m = \bar{\bar{f}}_m \quad (m = 1, 2, 3);$$

for $\zeta < 0$ when $m = 1, 2$

$$\frac{\partial^2 \bar{\bar{u}}_m}{\partial\zeta^2} - q_m^2 \bar{\bar{u}}_m = 0;$$

$$k_t^2 \frac{\partial}{\partial\zeta}\left(\frac{1}{k_t^2}\frac{\partial \bar{\bar{u}}_3}{\partial\zeta}\right) - \lambda^2 q_3^2 \bar{\bar{u}}_3 = -i(\lambda^2 - 1)\frac{\partial}{\partial\zeta}(\alpha\bar{\bar{u}}_1 + \beta\bar{\bar{u}}_2) + ik_t^2 \frac{\partial}{\partial\zeta}\left(\frac{1}{k_t^2}\right)(\alpha\bar{\bar{u}}_1 + \beta\bar{\bar{u}}_2);\ [\bar{\bar{u}}_m(\zeta_p)] = 0 \quad (m=1,2,3);$$

$$\left[\frac{\partial \bar{\bar{u}}_m}{\partial\zeta}(\zeta_p)\right] = 0\ (m = 1, 2);\ \left[\frac{1}{k_t^2}\left(-i\alpha\bar{\bar{u}}_1(\zeta_p) - i\beta\bar{\bar{u}}_2(\zeta_p) + \frac{\partial \bar{\bar{u}}_3}{\partial\zeta}(\zeta_p)\right)\right] = 0,$$

where

$$\bar{\bar{u}}_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_m(\xi, \eta, \zeta)\, e^{i(\alpha\xi + \beta\eta)}\, d\xi d\eta;$$

$$\bar{\bar{f}}_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_m(\xi, \eta, \zeta)\, e^{i(\alpha\xi + \beta\eta)}\, d\xi d\eta;$$

$$q_0 = \sqrt{\alpha^2 + \beta^2 - k_0^2};\ q_m = \sqrt{\alpha^2 + \beta^2 - k_m^2},\ \operatorname{Re} q_0, q_m > 0.$$

Let us introduce the function $v_m(\zeta) = \bar{\bar{u}}_m(\zeta) - w_m(\zeta)$, where w_m are the solutions of the problems for $\zeta > 0$

$$\frac{\partial^2 w_m}{\partial\zeta^2} - q_0^2 w_m = \bar{\bar{f}}_m \text{ for } w_m(0) = 0,\ w_m(-\infty) = 0\ (m = 1, 2);$$

$$\frac{\partial^2 w_3}{\partial\zeta^2} - q_3^2 w_3 = \bar{\bar{f}}_3 \text{ for } \frac{\partial w_3}{\partial\zeta}(0) = 0,\ w_3(-\infty) = 0.$$

The functions v_m are defined as the solutions of the following boundary problems:

$$\frac{\partial^2 v_m}{\partial\zeta^2} - q_0^2 v_m = 0\ (m = 1, 2, 3,) \text{ for } \zeta > 0; \tag{5}$$

$$\frac{\partial^2 v_m}{\partial\zeta^2} - q_m^2 v_m = 0\ (m = 1, 2) \text{ for } \zeta < 0;$$

$$k_t^2 \frac{\partial}{\partial\zeta}\left(\frac{1}{k_t^2}\frac{\partial v_3}{\partial\zeta}\right) - \lambda^2 q_3^2 v_3 = 0.$$

At the boundary $\zeta = 0$ when $m = 1, 2$ the function $v_m(\zeta)$ is continuous but its derivative undergoes a discontinuity,

$$[v_m(0)],\ [v_m'(0)] = -w_m'(0),$$

at the other interfaces for $\zeta = \zeta_p$ ($p = 1, 2, \ldots, n-1$) $v_m(\zeta)$ and $v'_m(\zeta)$ are continuous

$$[v_m(\zeta_p)] = [v'_m(\zeta_p)] = 0. \tag{6}$$

For $m = 3$ the function $v_3(\zeta)$ suffers a discontinuity at the boundary $\zeta = 0$

$$[v_3(0)] = -w_3(0), \tag{7}$$

while for $\zeta = \zeta_p$ ($p = 1, 2, \ldots, n-1$) $v_3(\zeta)$ is continuous

$$[v_3(\zeta_p)] = 0.$$

In addition to this for $\zeta = \zeta_p$ ($p = 0, 1, 2, \ldots, n-1$) we have

$$\left[(1/k_t^2)\left(-i\alpha v_1(\zeta_p) - i\beta v_2(\zeta_p) + v_3'(\zeta_p)\right)\right] = 0.$$

At infinity the functions $v_m(\zeta)$ ($m = 1, 2, 3$) tend to zero. Following [7], in place of the function $v_3(\zeta)$ we introduce the function $z(\zeta)$ by means of the equation

$$z(\zeta) = v_3(\zeta) - [i/(\alpha^2+\beta^2)]\left(\alpha\frac{\partial v_1}{\partial\zeta} + \beta\frac{\partial v^2}{\partial\zeta}\right). \tag{8}$$

For the function $z(\zeta)$ we then have the simpler boundary problem

$$\partial^2 z/\partial\zeta^2 - q_0^2 z = 0 \quad \text{for} \quad \zeta > 0, \tag{9}$$

$$k_t^2\frac{\partial}{\partial\zeta}\left(\frac{1}{k_t^2} - \frac{\partial z}{\partial\zeta}\right) - \lambda^2 q_3^2 z = 0 \quad \text{for} \quad \zeta < 0.$$

At the boundary $\zeta = 0$ we have the condition

$$[z(0)] = -w_3(0) + [i/(\alpha^2+\beta^2)]\left(\alpha\frac{\partial w_1}{\partial\zeta}(0) + \beta\frac{\partial w_2}{\partial\zeta}(0)\right),$$

$$\left[\frac{1}{k_t^2}\frac{\partial z}{\partial\zeta}(0)\right] = 0,$$

at the other boundaries, for $\zeta = \zeta_p$ ($p = 1, 2, \ldots, n-1$)

$$[z(\zeta_p)] = 0, \left[\frac{1}{k_t^2}\frac{\partial z}{\partial\zeta}(\zeta_p)\right] = 0. \tag{10}$$

Starting from the fact that for $\zeta \geq 0$ the functions $v_m(\zeta)$ ($m = 1, 2$) and $z(\zeta)$ take the form

$$v_m(\zeta) = A_m e^{-q_0\zeta}, \ z(\zeta) = A_3 e^{-q_0\zeta},$$

and their derivatives are, respectively,

$$\frac{\partial v_m}{\partial\zeta}(\zeta) = -q_0 v_m(\zeta), \ \frac{\partial z}{\partial\zeta}(\zeta) = -q_0 z(\zeta),$$

we may transform the boundary conditions at the boundary $\zeta = 0$

$$\frac{\partial v_m}{\partial\zeta}(-0) = -q_0 v_m(-0) + \frac{\partial w_m}{\partial\zeta}(0) \quad (m = 1,2); \tag{11}$$

$$q_0 z(-0) + \frac{k_0^2}{k_{t_1}^2}\frac{\partial z(-0)}{\partial\zeta} = q_0 w_3(0) - \frac{q_0}{\alpha^2+\beta^2}\left(\alpha\frac{\partial w_1}{\partial\zeta}(0) + \beta\frac{\partial w_2}{\partial\zeta}(0)\right). \tag{12}$$

The problem is thus reduced to finding the functions $v_m(\zeta)$ ($m = 1, 2$) and $z(\zeta)$ for $\zeta < 0$, so as to satisfy the ordinary differential equations (5) and (9) with boundary conditions (11) and (12) for $\zeta = 0$, and conditions (6) and (8) for $\zeta = \zeta_p$ ($p = 1, 2, \ldots, n-1$). At infinity the functions $v_m(\zeta)$ ($m = 1, 2$), $z(\zeta)$ tend to zero. Let us construct the asymptotic for this problem. To this end we make the substitution $\zeta = t\epsilon$ ($t \leq 0$) in the equation and boundary conditions

$$\partial^2 v_m/\partial t^2 + d^2 v_m = (\alpha^2+\beta^2)\varepsilon^2 v_m \qquad (m = 1, 2); \tag{13}$$

$$d^2\frac{\partial}{\partial t}\left(\frac{1}{d^2}\frac{\partial z}{\partial t}\right) + d^2 z = (\alpha^2+\beta^2)\varepsilon^2 z\lambda^2(t), \ -\mathrm{Re}\, d > 0; \tag{14}$$

$$\frac{\partial v_m(0)}{\partial t} = -q_0 v_m(0)\varepsilon + \frac{\partial w_m}{\partial\zeta}(0)\varepsilon, \tag{15}$$

$$[v_m(t_p)] = \left[\frac{\partial v_m}{\partial t}(t_p)\right] = 0 \quad (p = 1, 2, \ldots, n-1);$$

$$q_0 z(0) + \frac{k_0^2}{k_{t1}^2 \varepsilon} \frac{\partial z}{\partial t}(0) = q_0 w_3(0) - \frac{q_0 i}{\alpha^2+\beta^2}\left(\alpha \frac{\partial w_1}{\partial \zeta}(0) + \beta \frac{\partial w_2}{\partial \zeta}(0)\right), [z(t_p)] = \left[\frac{1}{k_t^2}\frac{\partial z}{\partial t}(t_p)\right] = 0 \; (p = 1, 2, \ldots, n-1). \quad (16)$$

We seek the solution to the problem (13)-(16) in the form of a series

$$v_m = \sum_{j=1}^{s} \varepsilon^j v_{mj} + R_{ms} \; (m=1, 2), \; z = \sum_{j=0}^{s} \varepsilon^j z_j + R_s, \quad (17)$$

where R_{ms} and R_s are the residues of the series.

Substituting Eqs. (17) into the equations and boundary conditions of problem (13)-(16), we obtain a set of equations and boundary conditions for the successive determination of $v_{m1}, v_{m2}, \ldots, z_0, z_1, \ldots$

$$\partial^2 v_{mj}/\partial t^2 + d^2 v_{mj} = (\alpha^2+\beta^2) v_{mj-2} \; (m = 1, 2), (j = 1, 2, \ldots, s), \quad (18)$$
$$(v_{mj} = 0 \text{ for } j \leqslant 0);$$

$$d^2 \frac{\partial}{\partial t}\left(\frac{1}{d^2}\frac{\partial z_j}{\partial t}\right) + d^2 z_j = (\alpha^2+\beta^2) z_{j-2} \lambda^2(t) \; (z_j = 0 \text{ for } j < 0); \quad (19)$$

$$\frac{\partial v_{mj}(0)}{\partial t} = -q_0 v_{mj-1}(0) + \frac{\partial w_m(0)}{\partial \zeta}, \quad (20)$$

$$\frac{\partial w_m}{\partial \zeta}(0) = 0 \text{ for } j > 1, [v_{mj}(t_p)] = \left[\frac{\partial v_{mj}}{\partial t}(t_p)\right] = 0$$
$$(p = 1, 2, \ldots, (n-1);$$

$$q_0 z_j(0) + \frac{k_0^2}{b d_1^2}\frac{\partial z_j(0)}{\partial t} = q_0 w_3(0) - \frac{q_0 i}{\alpha^2+\beta^2}\left(\alpha \frac{\partial w_1}{\partial \zeta}(0) + \beta \frac{\partial w_2}{\partial \zeta}(0)\right) \quad (21)$$

$$\left(w_3(0) = \frac{\partial w_1}{\partial \zeta}(0) = \frac{\partial w_2}{\partial \zeta}(0) = 0 \text{ for } j > 0\right),$$
$$[z_j(0)] = \left[\frac{1}{d^2}\frac{\partial z(t_p)}{\partial t}\right] = 0 \; (p = 1, 2, \ldots, n-1).$$

At infinity the functions v_{mj} and z_j tend to zero. For $t < 0$ the functions v_{mj} and z_j are found as the solutions of the differential equations (18) and (19) in which the right-hand sides are known, subject to the conditions (20) and (21). If we know v_{mj} for $\zeta < 0$, then $v_m(\zeta)$ may be determined for $\zeta > 0$ in the form

$$v_m(\zeta) = \sum_{j=1}^{s} (\varepsilon^j v_{mj}(0) + R_{ms}(0)) e^{-q_0 \zeta} \; (m = 1, 2). \quad (22)$$

If we know $z_j(\zeta)$ for $\zeta < 0$, the function $v_3(\zeta)$ is determined for $\zeta > 0$ on taking account of (7), (8), and (22):

$$v_3(\zeta) = \sum_{j=0}^{s}\left\{\varepsilon^j \left[z_j(0) + \frac{i}{\alpha^2+\beta^2}\frac{\partial}{\partial \zeta}(\alpha v_{1j+1}(0) + \beta v_{2j+1}(0)) - w_3(0) + R_s(0) + \frac{\partial R_{1s}}{\partial \zeta}(0) + \frac{\partial R_{2s}}{\partial s}(0)\right\} e^{-q_0 \zeta}.$$

Knowing $v_m(\zeta)$ $(m = 1, 2, 3)$, we may find $\bar{\bar{u}}_m = v_m(\zeta) + w_m(\zeta)$, and then, after executing an inverse Fourier transformation, we obtain the solution to the problem (1)-(4) for $\zeta \geq 0$

$$u_m(\xi, \eta, \zeta) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{s}(\varepsilon^j v_{mj}(0) + R_{ms}(0)) e^{-q_0 \zeta} + w_m(\zeta)\right\} e^{-i(\alpha\xi+\beta\eta)} d\alpha d\beta; \quad (23)$$

$$u_3'(\xi, \eta, \zeta) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \sum_{j=0}^{s}\left\{\varepsilon^j\left[z_j(0) + \frac{i}{\alpha^2+\beta^2}\frac{\partial}{\partial \zeta} \times \right.\right.\right.$$
$$\left.\left.\left. \times (\alpha v_{1j+1}(0) + \beta v_{2j+1}(0))\right] w_3(0) + R_{3s}(0)\right\} e^{-q_0 \zeta} + w_3(\zeta)\right\rangle e^{-i(\alpha\xi+\beta\eta)} d\alpha d\beta, \quad (24)$$

where

$$R_{3s}(0) = R_s(0) + \frac{\partial R_{1s}}{\partial \zeta}(0) + \frac{\partial R_{2s}}{\partial \zeta}(0).$$

Using Eqs. (5) and (9) [subject to boundary conditions (11), (12), (6), and (10)] and the definition of the asymptotic (18)-(21), we obtain the equations and boundary conditions for the residues R_{ms}, R_s

$$\frac{\partial^2 R_{ms}}{\partial \zeta^2} - q_m^2 R_{ms} = (\alpha^2+\beta^2)(v_{ms-1} + v_{ms}) \quad (25)$$
$$(m = 1,2), \; v_{ms} = 0 \text{ for } s < 1;$$

$$k_t^2 \frac{\partial}{\partial \zeta}\left(\frac{1}{k_t^2}\frac{\partial R_s}{\partial \zeta}\right) - q_3^2\lambda^2 R_s = (\alpha^2 + \beta^2)\lambda^2 (z_{s-1} + z_s), \tag{26}$$

$z_s = 0$ for $s < 0$, $R_{m0} = v_m$ minus the exact solution $(m = 1, 2)$;

$$\frac{\partial R_{ms}(0)}{\partial \zeta} + q_0 R_{ms}(0) = -q_0 v_{ms}(0), \tag{27}$$

$$[R_{ms}(\zeta_p)] = \left[\frac{\partial R_{ms}}{\partial \zeta}(\zeta_p)\right] = 0 \; (p = 1, 2, \ldots, n-1);$$
$$\frac{\partial R_s(0)}{\partial \zeta} + \frac{k_0^2}{k_{t1}^2} R_s(0) = 0, \tag{28}$$
$$[R_s(\zeta_p)] = \left[\frac{1}{k_t^2}\frac{\partial R_s}{\partial \zeta}(\zeta_p)\right] = 0 \; (p = 1, 2, \ldots, n-1).$$

In addition to this R_{ms}, R_s tend to zero at infinity. In order to substantiate the asymptotic so constructed, we must estimate the residues R_{ms} $(m = 1, 2, 3)$. First we obtain an a priori estimate of the actual approximations. For this purpose we use the energy method, and following [8] introduce the functional Hilbert space H_0 in the interval $(-\infty, 0)$ with the scalar product

$$(v, u) = \int_{-\infty}^{0} e^{-\nu\zeta} v\bar{u}\, d\zeta, \; (v, v) = \|v\|^2$$

and also the Hilbert space H with the scalar product

$$(v, u)_H = b^2 \int_{-\infty}^{0} e^{-\nu\zeta} v\bar{u}\, d\zeta + \int_{-\infty}^{0} e^{-\nu\zeta} \frac{\partial v}{\partial \zeta}\frac{\partial \bar{u}}{\partial \zeta}\, d\zeta, \tag{29}$$
$$(v, v)_H = \|v\|_H^2,$$

where ν is a certain constant $0 < \nu < b$; $e^{-\nu\zeta}$ is a weighting function.

The spaces H_0 and H may be considered as closures with respect to the norm $\| \;\|$, $\| \;\|_H$ of a set of functions defined on the semiaxis $[-\infty, 0]$, continuous (together with their first derivatives) and equal to zero outside a certain segment $[-a, 0]$.

We shall substantiate the system for the case of "model" problems.

Consider the equation

$$Lx \equiv x'' - b_2^2 x = h \tag{30}$$

with the boundary conditions

$$x'(0) + Ax(0) = f(0), \; [x(\zeta_p)] = [x'(\zeta_p)] = 0 \tag{31}$$
$$(p = 1, 2, \ldots, n-1), \; x(-\infty) = 0$$

and the equation

$$L_1 x \equiv k^2 \frac{\partial}{\partial \zeta}\left(\frac{1}{k^2}\frac{\partial x}{\partial \zeta}\right) - b_2^2 x = h_1 \tag{32}$$

with the boundary conditions

$$Ax(0) + Bx'(0) = f_1(0), \; [x(\zeta_p)] = [(1/k^2)x'(\zeta_p)] = 0 \tag{33}$$
$$(p = 1, 2, \ldots, n-1), \; x(-\infty) = 0.$$

The complex functions $b_2^2 = b_2^2(\zeta)$, $k^2 = k^2(\zeta)$ are piecewise continuous $|k^2(\zeta)| > 0$, $\zeta = \zeta_p$ $(p = 1, 2, \ldots, n-1)$ are their points of discontinuity. Let us denote

$$b_0^2 = \min_{\zeta}\left(\operatorname{Re} b_2^2 + \operatorname{Im} b_2^2\right) > 0. \tag{34}$$

The functions $h = h(\zeta)$, $h_1 = h_1(\zeta)$, $f = f(\zeta)$, $f_1 = f_1(\zeta)$ belong to the space H; A and B are constants Re A, Im $A \geq 0$,

$$\operatorname{Re}(\bar{B}/A) + \operatorname{Im}(\bar{B}/A) \geqslant 0.$$

By the solution to the problem (30), (31) and (32), (33) we mean a function $x = x(\zeta)$, continuous on the semiaxis $[-\infty, 0]$, having continuous first and second derivatives at every point except the points $\zeta = \zeta_p$, and at the points $\zeta = \zeta_p$ having limiting values of the first derivatives to the left and right. Equations (30) and (32) are satisfied at every point of the semiaxis $(-\infty, 0)$, except the points $\zeta = \zeta_p$.

LEMMA 1. If the function $x = x(\zeta) \in H$, the following estimate holds:

$$|x(\zeta_p)| \leqslant \left(e^{\nu\zeta_p/2}/\sqrt{\nu}\right)\|x'\| \ (p = 0, 1, 2, \ldots, n-1), \zeta_0 = 0. \tag{35}$$

The proof is based on the Cauchy–Bunyakovskii inequality

$$|x(\zeta)|^2 - \left|\int_{-\infty}^{\zeta} x'(t)\,dt\right|^2 = \left|\int_{-\infty}^{\zeta} e^{\frac{\nu t}{2}} e^{-\frac{\nu t}{2}} x'(t)\,dt\right| \leqslant \int_{-\infty}^{\zeta} e^{\nu t}\,dt \int_{-\infty}^{\zeta} e^{-\nu t}|x'(t)|^2\,dt \leqslant \frac{e^{\nu\zeta}}{\nu}\|x'\|^2,$$

whence Eq. (35) follows.

LEMMA 2. Let the function $x = x(\zeta)$ be a solution of Eq. (30) subject to boundary conditions (31). The estimate

$$\|x\|_H^2 \leqslant C\left(\|f\|_H^2/b_0^2 + \|h\|^2/b_0^4\right). \tag{36}$$

is then valid. The constant C does not depend on f, h, b_0.

Proof. Let us write down the expression for the real and imaginary parts of the quadratic form of the operator $L \equiv (Lx, \bar{x})$:

$$\mathrm{Re}(h, x) = \mathrm{Re} f(0)\bar{x}(0) - \mathrm{Re} A|x(0)|^2 + \nu\,\mathrm{Re}\int_{-\infty}^{0} e^{-\nu\zeta} x'\bar{x}\,d\zeta - \|x'\|^2\,\mathrm{Re}\sum_{p=1}^{n}\int_{\zeta_p}^{\zeta_{p-1}} b_2^2|x|^2 e^{-\nu\zeta}\,d\zeta; \tag{37}$$

$$\mathrm{Im}(h, x) = \mathrm{Im} f(0)\bar{x}(0) - \mathrm{Im} A|x(0)|^2 + \nu\,\mathrm{Im}\int_{-\infty}^{0} e^{-\nu\zeta} x'\bar{x}\,d\zeta - \mathrm{Im}\sum_{p=1}^{m}\int_{\zeta_p}^{\zeta_{p-1}} b_2^2|x|^2 e^{-\nu\zeta}\,d\zeta, \tag{38}$$

where $\zeta_0 = 0$, $\zeta_n = -\infty$.

Adding (37) and (38), we obtain

$$\|x'\|^2 + \sum_{p=1}^{n}\int_{\zeta_p}^{\zeta_{p-1}} |x|^2 e^{-\nu\zeta}\left(\mathrm{Re}\, b_2^2 + \mathrm{Im}\, b_2^2\right) d\zeta + |x(0)|^2 \times$$
$$\times(\mathrm{Re}\, A + \mathrm{Im}\, A) = \mathrm{Re} f(0)\bar{x}(0) + \mathrm{Im} f(0)\bar{x}(0) + \nu\left(\mathrm{Re}\int_{-\infty}^{0} e^{-\nu\zeta} \times\right.$$
$$\left.\times\, x'\bar{x}\,d\zeta + \mathrm{Im}\int_{-\infty}^{0} e^{-\nu\zeta} x'\bar{x}\,d\zeta\right) - \mathrm{Re}(h, x) - \mathrm{Im}(h, x).$$

Taking account of Eq. (34), rejecting the nonnegative terms on the left-hand side, we have

$$\|x'\|^2 + b_0^2\|x\|^2 \leqslant 2|f(0)|\cdot|x(0)| + 2\nu\left|\int_{-\infty}^{0} e^{-\nu\zeta} x'\bar{x}\,d\zeta\right| + 2|(h, x)|.$$

Applying inequality (35) to the first term on the right-hand side and then applying the elementary inequality

$$(FG) \leqslant \frac{1}{2}\left[(1/\delta^2)\|F\|^2 + \delta^2\|G\|^2\right] \tag{39}$$

to all the terms on the right-hand side, and taking $\nu = b_0/3$, we obtain

$$\|x'\|^2 + b_0\|x\|^2 \leqslant C\left(|f(0)|^2/b_0^2 + \|h\|^2/b_0^4\right),$$

and Eq. (36) follows from this if we allow for (29) and (35) in relation to the function f(0).

LEMMA 3. For the solution of Eq. (32), subject to boundary conditions (33), a valid estimate is

$$|x'(\zeta_p)| \leqslant C\left(e^{\nu\zeta_p/2}/\sqrt{\nu}\right)\left(\|h\| + \|x\|\, b_0^2\right) \quad (p = 0, 1, \ldots, n-1). \tag{40}$$

The constant C does not depend on h or b_0.

Proof. Consider the equations derived from (32) and (33):

$$x'(\zeta_{p-1}) = k_p^2 \sum_{j=1}^{n}\int_{\zeta_j}^{\zeta_{j-1}} \frac{1}{k^2}\left(h_1 + b_2^2 x\right) d\zeta, \ \zeta_0 = 0, \ \zeta_n = -\infty.$$

We have the inequality

$$|x'(\zeta_{p-1})| \leqslant |k_p^2| \sum_{j=p}^{n} \left| \int_{\zeta_j}^{\zeta_{j-1}} \frac{h_1}{k^2} d\zeta \right|^2 + b_0^4 \left| \int_{\zeta_j}^{\zeta_{j-1}} \frac{b_2^2 x}{b_0^2 k^2} d\zeta \right|^2.$$

To each term of the latter inequality we apply inequality (35); allowing for

$$C = |k_p^2| \max \left\{ \max_{\zeta} \left| \frac{1}{k^2(\zeta)} \right|^2, \quad \max_{\zeta} \left| \frac{b_2^2(\zeta)}{k^2(\zeta)\, b_0^2} \right| \right\},$$

we deduce Eq. (40).

LEMMA 4. Let $x = x(\zeta)$ be the solution to Eq. (32) subject to the boundary conditions (33). A valid estimate is then

$$\|x\|_H^2 \leqslant C \left(\left\| \frac{f_1}{A} \right\|_H^2 + \frac{\|h\|^2}{b_0^4} \right). \tag{41}$$

The constant C does not depend on A, h_1, f_1, b_0. We may write down the expression for the real and imaginary parts of the quadratic form of the operator $L_1 \equiv (L_1 x, \bar{x})$

$$\mathrm{Re}(h_1, x) = \mathrm{Re}\, \frac{\overline{f(0)}}{A} x'(0) - \mathrm{Re}\, \frac{\overline{B}}{A} |x'(0)|^2 + \mathrm{Re} \sum_{p=1}^{n-1} e^{-\nu\zeta_p} \bar{x}(\zeta_p)\, x'(\zeta_p) \left(1 - \frac{k_{p+1}^2}{k_p^2}\right) + \nu\, \mathrm{Re} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} \bar{x} x' d\zeta -$$

$$- \mathrm{Re} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x'|^2 d\zeta - \mathrm{Re} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x|^2 b_2^2 d\zeta + \mathrm{Re} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu t} \bar{x} x' M(\zeta)\, d\zeta,$$

$$M(\zeta) = k^2 \frac{\partial}{\partial \zeta} \left(\frac{1}{k^2} \right);$$

$$\mathrm{Im}(h_1, x)\, \mathrm{Im}\, \frac{\overline{f(0)}}{A} x'(0) - \mathrm{Im}\, \frac{\overline{B}}{A} |x'(0)| + \mathrm{Im} \sum_{p=1}^{n-1} e^{-\nu\zeta_p} \bar{x}(\zeta_p)\, x'(\zeta_p) \left(1 - \frac{k_{p+1}^2}{k_p^2}\right) + \nu\, \mathrm{Im} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} \bar{x} x' d\zeta -$$

$$- \mathrm{Im} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x'|^2 d\zeta - \mathrm{Im} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x|^2 b_2^2 d\zeta + \mathrm{Im} \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu t} \bar{x} x' M(\zeta)\, d\zeta.$$

Adding the resultant expressions, we obtain

$$\sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x'|^2 d\zeta + \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x|^2 \left(\mathrm{Re}\, b_2^2 + \mathrm{Im}\, b_2^2\right) d\zeta +$$

$$+ |x'(0)|^2 \left(\mathrm{Re}\, \frac{\overline{B}}{A} + \mathrm{Im}\, \frac{\overline{B}}{A} \right) = \mathrm{Re}\, \frac{\overline{f_1(0)}}{A} x'(0) + \mathrm{Im}\, \frac{\overline{f_1(0)}}{A} x'(0) +$$

$$+ \sum_{p=1}^{n} e^{-\nu\zeta_p} \left[\mathrm{Re}\, \bar{x}(\zeta_p)\, x'(\zeta_p) \left(1 - \frac{k_{p+1}^2}{k_p^2}\right) + \mathrm{Im}\, \bar{x}(\zeta_p)\, x'(\zeta_p) \times \right.$$

$$\left. \times \left(1 - \frac{k_{p+1}^2}{k_p^2}\right) \right] + \nu \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} \left(\mathrm{Re}\, \bar{x} x' + \mathrm{Im}\, \bar{x} x'\right) d\zeta +$$

$$+ \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} \left[\mathrm{Re}\, \bar{x} x' M(\zeta) + \mathrm{Im}\, \bar{x} x' M(\zeta)\right] d\zeta - \mathrm{Re}(h_1, x) - \mathrm{Im}(h_1, x).$$

Assuming that $\max_{\zeta} |M(\zeta)| < C$, remembering (34), and discarding nonnegative terms on the left-hand side, we have

$$\|x'\|^2 + b_0^2 \|x\|^2 \leqslant 2 \left| \frac{f_1(0)}{A} \right| |x'(0)| + 2C_1 \sum_{p=1}^{n-1} e^{-\nu\zeta_p} |x(\zeta_p)| \times$$

$$\times |x'(\zeta_p)| + 2(\nu + C) \sum_{p=1}^{n} \int_{\zeta_p}^{\zeta_{p-1}} e^{-\nu\zeta} |x|\, |x'|\, d\zeta + 2|(h_1, x)|.$$

Applying inequality (40) to the first and second terms on the right and then applying inequality (39) to all the terms on the right, taking $\nu = b_0/3$, we have

$$\|x'\|^2 + b_0^2\|x\|^2 \leqslant C\left(\left|\frac{f_1(0)}{A}\right|^2 b_0 + \frac{\|h_1\|^2}{b_0^2}\right),$$

from which we may deduce (41) if we allow for (29) and the estimate (35) in relation to the function $f_1(0)$.

Let us now estimate the v_{mj} and z_j approximations from (18) and (19). We shall consider that the constants C only depend on the parameters

$$|k_0^2|,\ \max_\zeta|\lambda^2(\zeta)|,\ \max_\zeta\left|\frac{M^2(\zeta)}{b^2}\right|.$$

THEOREM 1. In order to solve Eq. (18) (m = 1, 2) subject to the boundary conditions (20), and Eq. (19) subject to the boundary conditions (21), the following estimates are valid:

$$\|v_{mj}\|_H \leqslant \frac{C_{mj}}{b^{j-1/2}}\left|\frac{\partial w_m}{\partial\zeta}(0)\right| |q_0'|^{j-1} (m = 1, 2), \tag{42}$$

$$(j = 1, 2, \ldots, s),\ |q_0'| = \sqrt[4]{(\alpha^2+\beta^2+a_0^2)^2 + b_0^4};$$

$$\|z_j\|_H \leqslant \frac{C_{32j}}{b^{2j-1/2}}(\alpha^2+\beta^2)^j\left\{|w_3(0)| + \frac{1}{\alpha^2+\beta^2}\left(\left|\alpha\frac{\partial w_1}{\partial\zeta}(0)\right| + \left|\beta\frac{\partial w_2}{\partial\zeta}(0)\right|\right)\right\},\ j = 0, 1, 2, \ldots, s. \tag{43}$$

Proof. Applying Lemma 2 successively to Eq. (18) (m = 1, 2) subject to boundary conditions (20), where

$$x(\zeta) = v_{mj}(t),\ b_2^2(\zeta) = -d^2(t),\ h(\zeta) = (\alpha^2+\beta^2)v_{mj-2}(t),\ A = 0,$$
$$f(0) = -q_0 v_{mj-1} + \frac{\partial w_m}{\partial\zeta}(0),\ \frac{\partial w_m}{\partial\zeta}(0) = 0 \text{ for } j > 1,\ v_{mj} = 0,\ j \leqslant 0$$

for each j = 1, 2, ..., s, and using mathematical induction with respect to j, we obtain (42). Applying Lemma 4 to Eq. (19) subject to boundary condition (21), where $x(\zeta) = z_j(t)$,

$$b_2^2(\zeta) = -d^2(t),\ k^2(\zeta) = d^2(t);\ \ h(\zeta) = (\alpha^2+\beta^2)z_{j-2}\lambda^2(t),\ \ A = q_0,$$

$$B = k_0^2/bd_1^2,\ f_1(0) = q_0 w_3(0) - \frac{q_0 i}{\alpha^2+\beta^2}\left(\alpha\frac{\partial w_1}{\partial\zeta}(0) + \beta\frac{\partial w_2}{\partial\zeta}(0)\right),\ \ w_3(0) = \frac{\partial w_1}{\partial\zeta}(0) = \frac{\partial w_2}{\partial\zeta}(0) \text{ for } j > 0,$$

and proceeding in the same way as before, we obtain (43).

THEOREM 2. In order to solve Eq. (25) (m = 1, 2) subject to the boundary conditions (27) and Eq. (26) subject to the boundary conditions (28) we have the following valid estimates:

$$\|R_{ms}\|_H \leqslant C_{ms}\frac{|q_0'|^3}{b^{s+1/2}}\left|\frac{\partial w_m}{\partial\zeta}(0)\right| (m = 1, 2); \tag{44}$$

$$\|R_s\|_H \leqslant C_{32s}\frac{(\alpha^2+\beta^2)^{s+1}}{b^{3/2+2s}}\left(|w_3(0)| + \frac{1}{\alpha^2+\beta^2}\left(\left|\alpha\frac{\partial w_1}{\partial\zeta}(0)\right| + \left|\beta\frac{\partial w_2}{\partial\zeta}(0)\right|\right)\right) (s = 0, 1, 2, \ldots). \tag{45}$$

Proof. Applying Lemma 2 to Eq. (25) (m = 1, 2) subject to boundary condition (27), where

$$x(\zeta) = R_{ms}(\zeta);\ b_2^2(\zeta) = q_m^2(\zeta);\ h(\zeta) = (\alpha^2+\beta^2)(v_{ms-1} + v_{ms});\ A = q_0;$$
$$f(0) = -q_0 v_{ms}(0);\ v_{ms} = 0$$

for $s \leq 0$, $R_{m0} = v_m$ minus the exact solution, and allowing for (42), we obtain (44). Applying Lemma 4 to Eq. (26), subject to boundary condition (28), where

$$x(\zeta) = R_s(\zeta);\ b_2^2(\zeta) = q_3^2(\zeta)\lambda^2;\ h_1(\zeta) = (\alpha^2+b^2)\lambda^2(\zeta)(z_{s-1} + z_s);$$
$$A = q_0;\ B = k_0^2/k_{t_1}^2;\ k^2(\zeta) = k_t^2(\zeta);\ z_s = 0 \text{ for } s < 0,\ f_1(0) = 0,$$

and allowing for estimate (43), we obtain (45).

THEOREM 3 (this is proved on the basis of the theorems already obtained). For the residual term

$$\widetilde{R}_{ms} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ms}(0)\,e^{-q_0\zeta}e^{-i(\alpha\xi+\beta\eta)}\,d\alpha d\beta \tag{46}$$
$$(m = 1, 2, 3),\ (s = 0, 1, 2 \ldots)$$

of the solution of (1)-(4) for $\zeta \geq 0$ a valid estimate is

$$|R_{ms}| \leqslant C_{ms}/b^{s+1}\ (m = 1, 2, 3),\ (s = 0, 1, \ldots). \tag{47}$$

Proof. The validity of estimate (47) follows from estimates (35), (44), and (45) and the bounded nature of the integrals

$$\int_0^\infty |q_0'|^s e^{-q_0}|\gamma d\gamma < C, \int_0^\infty |q_0'|^s |e^{-q_0}| d\gamma < C, \int_0^\infty \frac{\gamma^{s+2}}{|q_0|} |e^{-q_0}| d\gamma < C,$$
$$|q_0'| = \sqrt[4]{(\gamma^2 + a_0^2)^2 + b_0^4}, \quad |q_0| = \sqrt[4]{(\gamma^2 - a_0^2)^2 + b_0^4}.$$

These latter integrals are obtained if in (46) we make the substitution

$$\alpha = \gamma \cos \varphi, \ \beta = \gamma \sin \varphi, \ \xi = \rho \cos \psi, \ \eta = \rho \sin \psi.$$

Before proving the solubility of problems (18)-(21), (25)-(28), the solutions of which are used in constructing the asymptotic, let us consider an analogous problem for the case in which the lower medium is an ideal conductor, i.e., for $\zeta = \zeta_n$, $1/k^2(\zeta_n) = 0$ $(\zeta_n < \zeta_{n-1})$.

We consider the following problem A: to find a solution of Eqs. (1)-(3) for $\zeta > \zeta_n$, satisfying the boundary conditions (4), and for $\zeta = \zeta_n$ satisfying the conditions

$$u_1 = u_2 = 0, \ du_3/d\zeta = 0.$$

The asymptotic for the solution of problem A may be constructed in the same way as for problem (1)-(4). In order to estimate the approximations and residues we may furthermore make use of Lemmas 1-4 if we define the function $v_{mj} = R_{mj} = 0$ $(m = 1, 2)$ completely for $\zeta < \zeta_n$ and the functions $z_j R_s$ for $\zeta < \zeta_n$ so that they belong to the space H. Theorems 1 and 2 may therefore also be regarded as proved for problem A, the constants C_{mj} not depending on ζ_n.

The solubility of the corresponding equations (18)-(21) and (25)-(28) for v_{mj}, z_j, R_{ms}, R_s in the case of problem A follows from the estimates (42)-(45), since v_{mj}, z_j, R_{ms}, R_s are the solutions of ordinary linear differential equations in each of the segments (ζ_j, ζ_{j+1}) with continuous coefficients. If we make ζ_n tend to $-\infty$, we deduce the solubility of the corresponding equations for the $v_{mj} - R_s$ of the original problem.

As an example of the application of the asymptotic so constructed, we may give the solution to problem (1)-(4) for the upper half-space when the function $k_t^2(\zeta)$ is piecewise constant (each layer has a constant conductivity). In accordance with (23) and (24) we have

$$u_m = w_m + \frac{1}{b} u_{1m} + \frac{1}{b^2} u_{2m} + \widetilde{R}_{m2} = \frac{\mu_0 I_m}{8\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{q_0} \times$$
$$\times \left(e^{-q_0|1+\zeta|} - e^{-q_0(\zeta+1)}\right) e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)} d\alpha d\beta +$$
$$+ \frac{\mu_0 I_m}{4\pi^2} \frac{iR^*}{k_{t_1}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-q_0(1+\zeta)} e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)]} d\alpha d\beta -$$
$$- \frac{\mu_0 I_m}{4\pi^2} \frac{R^{*2}}{k_{t_1}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-q_0(1+\zeta)} q_0 e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1]} d\alpha d\beta + \widetilde{R}_{m_2}; \tag{48}$$

$$u_3 = w_3 + u_{30} + \frac{1}{b} u_{31} + \widetilde{R}_{31} = \frac{\mu_0 I_3}{8\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{q_0} \left(e^{-q_0|\zeta-1|} - \right.$$
$$\left. - e^{-q_0(\zeta+1)}\right) e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)]} d\alpha d\beta + \frac{\mu_0 I_3}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-q_0(1+\zeta)} e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)]}}{q_0 - \frac{ik_0^2}{k_{t_1} R_*}} \times$$
$$\times d\alpha d\beta + \frac{\mu_0}{4\pi^2} \frac{k_0^2}{k_{t1} R_*} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\alpha I_1 + \beta I_2) e^{-q0(1+\zeta)} e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)]}}{(a^2+\beta^2)\left(q_0 - \frac{ik_0^2}{k_{t_1} R_*}\right)} \times$$
$$\times d\alpha d\beta + \frac{\mu_0}{4\pi^2} \frac{R^*}{k_{t_1}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\alpha I_1 + \beta I_2)}{\alpha^2+\beta^2} q_0 e^{-q_0(1+\zeta)} e^{-i[\alpha(\xi-\xi_1)+\beta(\eta-\eta_1)} d\alpha d\beta + \widetilde{R}_{31}, \tag{49}$$

where following [7]

$$R^* = \operatorname{cth}\left[ik_{t1}h_1 + \operatorname{arcth}\frac{k_{t1}}{k_{t_2}} \operatorname{cth}\left(ik_{t_2}h_2 + \ldots + \operatorname{arcth}\left(-\frac{k_{tn-1}}{k_{tn}}\right)\right)\right];$$
$$R_* = \operatorname{cth}\left[ik_{t1}h_1 + \operatorname{arcth}\frac{k_{t2}}{k_{t1}} \operatorname{cth}\left(ik_{t_2}h_2 + \ldots + \operatorname{arcth}\left(-\frac{k_{tn}}{k_{tn-1}}\right)\right)\right],$$
$$h_i = \zeta_{i+1} - \zeta_i.$$

We see from Eqs. (48) and (49) that for large b the dependence of the field in the upper medium on the parameters of the lower media is the same for any direction of the radiator. The resultant asymptotic may be used in solving geophysical problems [7].

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